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A STRUCTURE THEOREM FOR THE ELEMENTARY UNIMODULAR VECTOR GROUP

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Abstract. Given a pair of vectors $v, w \in R^{r+1}$ with $\langle v, w\rangle = v \cdot w^T = 1$, A. Suslin constructed a matrix $S_r(v, w) \in Sl_{2^r}(R)$. We study the subgroup $SUm_r(R)$ generated by these matrices, and its (elementary) subgroup $EUm_r(R)$ generated by the matrices $S_r(e_1\varepsilon, e_1{\varepsilon^T}^{-1})$, for $\varepsilon \in E_{r+1}(R)$. The basic calculus for $EUm_r(R)$ is developed via a key lemma, and a fundamental property of Suslin matrices is derived.

1. Introduction

In his doctoral thesis A. Suslin showed that a unimodular vector of the form $(a_0, a_1, a_2^2, \cdots, a_r^r)$ can be completed to an invertible matrix (*cf.* [4, Proposition 1.6]). In the beautiful and thought provoking section §5 of this paper he gives an inductive (on r) method of constructing the completion. Given a pair of vectors $v, w \in M_{1\ r+1}(R)$ he defines a matrix $S_r(v, w) \in M_{2^r}(R)$, with determinant equal to the inner product $v.w^T$. Consequently, if the dot product is 1, then $S_r(v, w) \in Sl_{2^r}(R)$. The actual completion is shown to be in the class $S_r(v, w)E_{2^r}(R)$. (For any n, $E_n(R)$ will denote the elementary subgroup of $Sl_n(R)$ generated by the elementary generators $E_{ij}(\lambda)$, $\lambda \in R$, $1 \le i \ne j \le n$, which is the matrix with 1's on the diagonal, the (i, j)-th entry λ, and the rest of the entries to be zero. An element of $E_n(R)$ will be called an elementary matrix.)

The Suslin matrices have proved useful in several contexts. We mention some examples. (Only the first and third results mentioned below have been published so far.)

• A. Suslin showed that, for a field k,

$$SK_1\left(\frac{k[x_1, \cdots, x_n, y_1, \cdots, y_n]}{(\sum_{i=1}^n x_i y_i - 1)}\right) \simeq \mathbb{Z},$$

with generator $[S_{n-1}((x_1, \cdots, x_n), (y_1, \cdots, y_n))]$.

• Let $\sum_{i=1}^n x_i y_i = 1$. Let P, P^* be the projective modules corresponding to the unimodular rows $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, respectively. Then $P^* \simeq Hom_R(P, R)$, the dual of P. If n is even, then $P \simeq P^*$. However, if $n > 1$ is odd, then M.V. Nori, and R.G. Swan independently showed (using topological arguments) that P, P^* need not be isomorphic. This can also be shown using the Suslin matrices, when $n > 3$.

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• Inspired by the proof of Serre's conjecture on the freeness of projective modules over a polynomial extension $k[x_1, \cdots, x_n]$ of a field k, and N. Mohan Kumar's theorems on set theoretic complete intersections, M. Boratýnski showed that any ideal I in a polynomial ring R over a field can be generated up to radical by $m = \mu(I/I^2)$ elements. Here μ denotes the minimal number of generators of the R/I-module I/I^2. Thus, one has $\sqrt{I} = \sqrt{(f_1, \cdots, f_m)}$, for some $f_1, \cdots, f_m \in R$.

• The orbit spaces $Um_n(R)/E_n(R)$ have a nice Witt group structure, if R is a noetherian ring of Krull dimension d, in which, say, -1 is a square, and if $n \geq \max\{3, (d+3)/2\}$.

The key question raised by A. Suslin in [4] was whether these matrices had a role in defining a symbol on the orbits space $Um_n(R)/E_n(R)$, similar to the role of the Vaserstein symbol defined in [3], when $n = 3$ and $\dim R = 2$. This has surfaced partially in the last application mentioned above, due to the study initiated in this paper.

This paper arose out of '*a desire to play with the Suslin matrices*'. Due to the inspiring [4, §5] there are several ways in which one can proceed to do this. We choose one such natural way in this article. With this aim, we introduce and study the subgroup $SUm_r(R)$ of $Sl_{2^r}(R)$, which we christen the Special Unimodular Vector group, generated by the Suslin matrices $S_r(v, w)$, corresponding to the pairs (v, w), with $v.w^T = 1$. Analogous to the elementary subgroup $E_n(R)$ we consider the Elementary Unimodular Vector subgroup $EUm_r(R)$, of $SUm_r(R)$, generated by the matrices $S_r(v, w)$ corresponding to the pair (v, w), with $v.w^T = 1$, and with v the first row of an elementary matrix. We develop the basic calculus for this group. As a consequence, we realize an important fundamental property of the Suslin matrices. (We exploit this property, and the equivalent key lemma here, to study the orbit space of unimodular vectors under elementary action, in a subsequent article.)

2. Elementary orbit equals Cohn orbit

Let R be a commutative ring with identity 1. Let $Um_{r+1}(R)$ denote the set of all unimodular vectors $v = (a_0, a_1, \cdots, a_r)$, i.e. $\sum a_i b_i = 1$, for some $b_i \in R$, $0 \leq i \leq r$.

The vector $w = (b_0, b_1, \cdots, b_r)$ is said to be *related* to v if the dot product $v.w^T = 1$. Note that vectors related to the same vector lie in the same elementary orbit, if their length is at least 3: If $v.w_1^T = 1 = v.w_2^T$, then $\varepsilon = I_{r+1} + v^T(w_2 - w_1) \in E_{r+1}(R)$ by [5, Corollary 2.7], and $w_1 \varepsilon = w_2$.

Definition. Let $v = (a_0, a_1, \cdots, a_r)$ and $w = (b_0, b_1, \cdots, b_r)$ with $v.w^T = 1$. We say that the vector

$$v^* = vC_{ij}(\lambda) = (a_0, \cdots, a_i + \lambda b_j, \cdots, a_j - \lambda b_i, \cdots, a_r),$$

for $0 \leq i \neq j \leq r$, is a *Cohn transform* of v w.r.t. the (related) vector w. One could write $C_{ij}(w, \lambda)$ if one wishes to specify w, but we generally refrain from doing so, as the related vector being considered is clear from the context. We shall say that a vector v^* is in the Cohn orbit of v if there is a related vector w^* to v^*, and a sequence of pairs, starting with $(v_0, w_0) = (v, w)$ and ending with $(v_r, w_r) = (v^*, w^*)$, such that, for $i \geq 0$, the pairs (v_{i+1}, w_{i+1}) have either v_{i+1} as a Cohn transform of v_i w.r.t. w_i, and $w_{i+1} = w_i$; or w_{i+1} as a Cohn transform of w_i w.r.t. v_i, and

$v_{i+1} = v_i$:

$$(v, w) = (v_0, w_0) \to (v_1, w_1) \to \cdots \to (v_r, w_r) = (v^*, w^*).$$

It follows inductively, by the above observation, that $v^* \in vE_{r+1}(R)$, i.e. Cohn transforms lie in the same elementary orbit. Conversely, any elementary transformation of a vector can be obtained by means of a sequence of Cohn transforms w.r.t. some suitable related vectors. We show this explicitly below in a special case, when $r = 2$. The general case can be easily reduced to the case when $r = 2$, and we leave this for the reader to rediscover for himself.

Let $aa' + bb' + cc' = 1$. We show that $(a, b, c) \longrightarrow (a + \lambda b, b, c)$ can obtain a sequence of Cohn transforms. The idea comes from analysing the proof of [3, Lemma 5.1]:

$$\begin{aligned}
\{(a,b,c);(a',b',c')\} &\overset{C_{02}(-\lambda)}{\longrightarrow} \{(a-\lambda c', b, c+\lambda a');\ (a',b',c')\}\\
&\overset{C_{12}(-1)}{\longrightarrow} \{(a-\lambda c', b, c+\lambda a');\ (a', b'-(c+\lambda a'), c'+b)\}\\
&\overset{C_{02}(\lambda)}{\longrightarrow} \{((a-\lambda c')+\lambda(c'+b), b, (c+\lambda a')-\lambda a');\\
&\qquad (a', b'-(c+\lambda a'), c'+b)\}\\
&= \{(a+\lambda b, b, c);(a', b'-c-\lambda a', c'+b)\}\\
&\overset{C_{12}(1)}{\longrightarrow} \{(a+\lambda b, b, c);(a', b'-\lambda a', c')\}.
\end{aligned}$$

Thus, one proves that

Lemma 2.1. *The elementary orbit $vE_{r+1}(R)$ of $v \in Um_{r+1}(R)$ coincides with the Cohn orbit of v, for $r \geq 2$.* □

3. The unimodular vector groups

We begin by describing the inductive process by which the Suslin matrix $S_r(v, w)$ of size 2^r, and of determinant $v \cdot w^T$, is constructed from two vectors v, w of size $r + 1$. We recall this process.

Let $v = (a_0, v_1)$, $w = (b_0, w_1)$. Set $S_0(v, w) = a_0$, and set

$$S_r(v,w) \quad = \quad \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1,w_1) \\ -S_{r-1}(w_1,v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}.$$

In [4, Lemma 5.1] it is noted that

1. $S_r(v, w)S_r(w, v)^T = (v.w^T)I_{2^r} = S_r(w, v)^T S_r(v, w)$, and
2. $\det S_r(v, w) = (v.w^T)^{2^{r-1}}$, for $r \geq 1$.

A. Suslin then describes a sequence of forms $J_r \in M_{2^r}(R)$ by the recurrence formulae

$$J_r = \begin{cases} 1, & \text{for } r = 0,\\ J_{r-1} \perp -J_{r-1}, & \text{for } r \text{ even},\\ J_{r-1} \top - J_{r-1}, & \text{for } r \text{ odd}. \end{cases}$$

(The English translation wrongly says $J_r = J_{r-1} \perp J_{r-1}$ when r is even.)

(Here, for $\alpha \in M_{r,s}(R)$, $\beta \in M_{p,q}(R)$, $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, while $\alpha \top \beta = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in M_{r+p,s+q}(R)$.)

It is easy to see that $\det J_r = 1$, for all r, and that $J_r^T = J_r^{-1} = (-1)^{\frac{r(r+1)}{2}} J_r$. Moreover, J_r is antisymmetric if $r = 4k+1$ and $r = 4k+2$, whereas J_r is symmetric for $r = 4k$ and $r = 4k+3$. In [4, Lemma 5.3], it is noted that the following formulae (called the **Suslin identities**) are valid:

$$\begin{aligned}
\text{for } r = 4k : (S_r(v,w)J_r)^T &= S_r(v,w)J_r;\\
\text{for } r = 4k+1 : S_r(v,w)J_rS_r(v,w)^T &= (v.w^T)J_r;\\
\text{for } r = 4k+2 : (S_r(v,w)J_r)^T &= -S_r(v,w)J_r;\\
\text{for } r = 4k+3 : S_r(v,w)J_rS_r(v,w)^T &= (v.w^T)J_r.
\end{aligned}$$

Definition. The *Special Unimodular Vector group* $SUm_r(R)$ is the subgroup of $Sl_{2^r}(R)$ generated by the Suslin matrices $S_r(v,w)$ w.r.t. the pair (v,w), with $v \in Um_{r+1}(R)$, and for some w related to v, i.e. $w.v^T = 1$.

The *Elementary Unimodular Vector group* $EUm_r(R)$ is the subgroup of $SUm_r(R)$ generated by the Suslin matrices $S_r(v,w)$, with $v \in e_1E_{r+1}(R)$, and with $w.v^T = 1$.

We shall denote by $EUm_r(R)^*$ the subgroup of $EUm_r(R)$ generated by the elements $S_r(e_1 + \lambda e_i, e_1)$, $S_r(e_1, e_1 + \lambda e_i)$, with $1 < i \leq r+1$, $\lambda \in R$.

Let us first note a few preliminary observations.

Lemma 3.1. *Let R be a commutative ring and let $v,\ w,\ s,\ t \in R^{r+1}$. Let $v = (a_0, a_1, \cdots, a_r)$, $w = (b_0, b_1, \cdots, b_r)$. Then*

$$\begin{aligned}
S_r(v,w) + S_r(w,v)^T &= \{a_0 + b_0\}I_{2^r},\\
S_r(s,t)S_r(w,v)^T + S_r(v,w)S_r(t,s)^T &= \{\langle s,w\rangle + \langle v,t\rangle\}I_{2^r},\\
S_r(w,v)^TS_r(s,t) + S_r(t,s)^TS_r(v,w) &= \{\langle s,w\rangle + \langle v,t\rangle\}I_{2^r}.
\end{aligned}$$

Proof. One has an inner product on $R^{r+1} \times R^{r+1}$ defined by $\langle v,w\rangle = v.w^T$. Also one has the usual bilinear consequence of the quadratic relation

$$\langle v,w\rangle I_{2^r} = S_r(v,w)S_r(w,v)^T = S_r(w,v)^TS_r(v,w),$$

viz.

$$\begin{aligned}
S_r(v+s, w+t)S_r(w+t, v+s)^T &= \langle v+s, w+t\rangle I_{2^r}\\
&= \{\langle v,w\rangle + \langle v,t\rangle + \langle s,w\rangle + \langle s,t\rangle\}I_{2^r}.
\end{aligned}$$

But

$$\begin{aligned}
&S_r(v+s, w+t)S_r(w+t, v+s)^T\\
&\quad= \{S_r(v,w) + S_r(s,t)\}\{S_r(w,v)^T + S_r(t,s)^T\}\\
&\quad= S_r(v,w)S_r(w,v)^T + S_r(v,w)S_r(t,s)^T + S_r(s,t)S_r(w,v)^T + S_r(s,t)S_r(t,s)^T\\
&\quad= \langle v,w\rangle I_{2^r} + S_r(v,w)S_r(t,s)^T + S_r(s,t)S_r(w,v)^T + \langle s,t\rangle I_{2^r}.
\end{aligned}$$

Equating the above two equations, we get

$$S_r(s,t)S_r(w,v)^T + S_r(v,w)S_r(t,s)^T = \{\langle s,w\rangle + \langle v,t\rangle\}I_{2^r}.$$

Similarly, one can prove the third relation. The first relation is obtained from the second by considering $s = t = e_1$. □

Notation. For a matrix $\alpha \in M_k(R)$, we define α^{top} as the matrix whose entries are the same as those of α above the diagonal, and on the diagonal, and is zero below the diagonal. Similarly, we define α^{bot}.

For simplicity we may write α^t for α^{top}, α^b for α^{bot}, and α^T for α transpose. Moreover, we use α^{tb} for α^{top} or α^{bot}. Similarly, we write $EUm_r(R)^{tb}$ to be the subgroup of $E_{2^r}(R)$ generated by the elements α^{tb}, for α a basic generator of $EUm_r(R)^*$.

We also write expressions like $x\alpha^{tb}x^{-1} = \beta^{bt}$ to mean that both $x\alpha^{top}x^{-1} = \beta^{bot}$ and $x\alpha^{bot}x^{-1} = \beta^{top}$ hold.

We now prove the key lemma.

Lemma 3.2. *Let $v = (a_0, a_1, \cdots, a_r) = (a_0, v_1)$, $w = (b_0, b_1, \cdots, b_r) = (b_0, w_1)$, with $v.w^T = 1$. Then, for $2 \le i \le r+1$, $r \ge 2$,*

$$\begin{aligned} S_r(e_1, e_1 + \lambda e_i)^{top} S_r(v,w) S_r(e_1, e_1+\lambda e_i)^{bot} &= S_r(vE_{i1}(-\lambda), wE_{1i}(\lambda)),\\ S_r(e_1+\lambda e_i, e_1)^{bot} S_r(v,w) S_r(e_1+\lambda e_i, e_1)^{top} &= S_r(vE_{1i}(\lambda), wE_{i1}(-\lambda)). \end{aligned}$$

(The case when $r = 1$ is similar.) Moreover, if $1 \le i \le r$, then

$$\begin{aligned} S_r(e_1+\lambda e_{i+1}, e_1)^{top} S_r(v,w) S_r(e_1+\lambda e_{i+1}, e_1)^{bot} &= S_r(vC_{0i}(-\lambda), w),\\ S_r(e_1, e_1+\lambda e_{i+1})^{bot} S_r(v,w) S_r(e_1, e_1+\lambda e_{i+1})^{top} &= S_r(v, wC_{0i}(-\lambda)). \end{aligned}$$

Proof. As $S_r(v,w)^{-1} = S_r(w,v)^T$, one needs to verify the first and third of the above identities, in order to deduce the other two identities. Both these identities are derived similarly, by means of a direct computation. We derive them as follows:

$$S_r(e_1, e_1+\lambda e_i)^{top} S_r(v,w) S_r(e_1, e_1+\lambda e_i)^{bot} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

where

$$\begin{aligned} \alpha_{11} &= a_0 I_{2^{r-1}} - S_{r-1}(0, \lambda e_{i-1}) S_{r-1}(w_1, v_1)^T - S_{r-1}(v_1, w_1) S_{r-1}(\lambda e_{i-1}, 0)^T \\ &\quad - b_0 S_{r-1}(0, \lambda e_{i-1}) S_{r-1}(\lambda e_{i-1}, 0)^T,\\ \alpha_{12} &= S_{r-1}(v_1, w_1) + b_0 S_{r-1}(0, \lambda e_{i-1}),\\ \alpha_{21} &= -S_{r-1}(w_1, v_1)^T - b_0 S_{r-1}(\lambda e_{i-1}, 0)^T,\\ \alpha_{22} &= b_0 I_{2^{r-1}}. \end{aligned}$$

Let $w^* = wE_{1i}(\lambda) = (b_0, w_1^*)$. Hence, via Lemma 3.1, the above matrix equals

$$\begin{pmatrix} (a_0 - \lambda a_{i-1}) I_{2^{r-1}} & S_{r-1}(v_1, w_1^*) \\ -S_{r-1}(w_1^*, v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix} = S_r(vE_{i1}(-\lambda), wE_{1i}(\lambda)),$$

as required. We now prove the third identity.

$$S_r(e_1+\lambda e_{i+1}, e_1)^{top} S_r(v,w) S_r(e_1+\lambda e_{i+1}, e_1)^{bot} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

where

$$\begin{aligned} \alpha_{11} &= a_0 I_{2^{r-1}} - S_{r-1}(\lambda e_i, 0) S_{r-1}(w_1, v_1)^T - S_{r-1}(v_1, w_1) S_{r-1}(\lambda e_i, 0) \\ &\quad - S_{r-1}(\lambda e_i, 0) S_{r-1}(0, \lambda e_i)^T,\\ \alpha_{12} &= S_{r-1}(v_1, w_1) + b_0 S_{r-1}(\lambda e_i, 0),\\ \alpha_{21} &= -S_{r-1}(w_1, v_1)^T - b_0 S_{r-1}(0, \lambda e_i)^T,\\ \alpha_{22} &= b_0 I_{2^{r-1}}. \end{aligned}$$

Let $v_1^* = (a_1, \cdots, a_i + \lambda b_0, \cdots, a_r)$. Then,

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} (a_0 - \lambda b_i) I_{2^{r-1}} & S_{r-1}(v_1^*, w_1) \\ -S_{r-1}(w_1, v_1^*)^T & b_0 I_{2^{r-1}} \end{pmatrix}.$$

This equals $S_r(vC_{0i}(-\lambda), w)$, as required. □

Remark. Note that we shall always regard $v = (a_0, a_1, \cdots, a_r) \in M_{1r+1}(R)$, as $\sum_{i=0}^{r} a_i e_{i+1}$. Thus, for instance, $e_1 + \lambda e_2 = (1, \lambda, \cdots, 0)$, etc. It will be convenient to use the following notation later: For $1 \le i \le r+1$, $\lambda \in R$, λ unit if $i = 1$,

$$\begin{aligned} E(e_1)(\lambda) &= \lambda I_{2^{r-1}} \perp \lambda^{-1} I_{2^{r-1}}, \\ E(e_i)(\lambda) &= S_r(e_1 + \lambda e_i, e_1);\ i > 1, \\ E(e_i^*)(\lambda) &= S_r(e_1, e_1 + \lambda e_i);\ i > 1. \end{aligned}$$

(If we wish to stress the size we could write $E_r(e_i)(\lambda)$, $E_r(e_i^*)(\lambda)$.)

Note also that $E(e_i)(\lambda)^T = E(e_i^*)(-\lambda)$ and $E(e_i^*)(\lambda)^T = E(e_i)(-\lambda)$ hold due to the "anti-symmetry" of the Suslin matrices.

The key lemma leads one to the fundamental property of Suslin matrices, via the methods of commutative algebra.

Corollary 3.3. *Let R be a commutative ring in which 2 is invertible. Let $\alpha = S_1 S_2 \cdots S_n$ and $\alpha^* = S_n S_{n-1} \cdots S_1$, where each S_i is a Suslin matrix $S_r(v_i, w_i)$, with $\langle v_i, w_i \rangle = 1$. For any $\beta = S_r(v, w)$, $\alpha\beta\alpha^* = S_r(v', w') \in SUm_r(R)$.*

Proof. Let $\wp$ be a prime ideal of R. Let $S_i = S_r(v_i, w_i)$, for $v_i, w_i \in R^{r+1}$, $\langle v_i, w_i \rangle = v_i . w_i^T = 1$. Note that $Um_{r+1}(R_\wp) = e_1 E_{r+1}(R_\wp)$. Using the observation at the beginning of Section 2, one can conclude that there is a $\varepsilon \in E_{r+1}(R_\wp)$ with $e_1\varepsilon = v$, and with $e_1 \varepsilon^{T^{-1}} = w$. Each ε_i can be written as a product of elementary generators of type $E_{1j}(\lambda)$, $E_{j1}(\mu)$, for $2 \le j \le r+1$, $\lambda, \mu \in R$. Hence, by Lemma 3.2,

$$S_r(v_i, w_i) \quad = \quad S_r(e_1\varepsilon_i, e_1\varepsilon_i^{T^{-1}}) = \sigma_{1i}^{tb} \cdots \sigma_{l_i i}^{tb} I_{2^r} \sigma_{l_i i}^{bt} \cdots \sigma_{1i}^{bt},$$

where σ_{pq}^{tb} is a generator of type $E(c)(\lambda)^{tb}$, for $c = e_t$ or e_t^*, $\lambda \in R$.

Observe that over $R_\wp$,

$$\begin{aligned} S_r(v_i, w_i) S_r(x, y) S_r(v_i, w_i) &= \sigma_{1i}^{tb} \cdots \sigma_{l_i i}^{tb} \sigma_{l_i i}^{bt} \cdots \sigma_{1i}^{bt} S_r(x, y) \sigma_{1i}^{tb} \cdots \sigma_{l_i i}^{tb} \sigma_{l_i i}^{bt} \cdots \sigma_{1i}^{bt} \\ &= S_r(x', y'), \end{aligned}$$

by Lemma 3.2, with $\langle x', y' \rangle = 1$. From this it follows that $\alpha\beta\alpha^*$ "locally looks like" an $S_r(v', w')$ with $\langle v', w' \rangle = 1$.

Hence, $\alpha\beta\alpha^*$ has a certain configuration amongst its entries. The reader should be able to convince him/herself that indeed $\alpha\beta\alpha^* = S_r(v', w')$, for some v', w' with $\langle v', w' \rangle = 1$. □

Remark 3.4. A direct proof of the fundamental property can be obtained via Lemma 3.1. We have included this in [2], where the property is studied in greater detail.

The usual elementary generators $E_{ij}(\lambda)$, $i \neq j$, $\lambda \in R$, of the elementary subgroup of the linear group, and the elementary symplectic generators $S_{ij}(\lambda)$, $i \neq j$, $\lambda \in R$, have certain "*splitting properties*". These are reflected in the basic elementary generators of $EUm_r(R)^*$, *viz.*

Lemma 3.5. *Let R be a commutative ring. For $2 \le i \le r+1$, $\lambda, \mu \in R$,*

$$\begin{aligned} E(e_i)(\lambda + \mu) &= E(e_i)(\lambda) E(e_i)(\mu), \\ E(e_i^*)(\lambda + \mu) &= E(e_i^*)(\lambda) E(e_i^*)(\mu), \\ E(e_i)(\lambda) &= E(e_i)(\lambda)^{top} E(e_i)(\lambda)^{bot} = E(e_i)(\lambda)^{bot} E(e_i)(\lambda)^{top}, \\ E(e_i^*)(\lambda) &= E(e_i^*)(\lambda)^{top} E(e_i^*)(\lambda)^{bot} = E(e_i^*)(\lambda)^{bot} E(e_i^*)(\lambda)^{top}. \end{aligned}$$

Proof. These will follow by a direct computation using Lemma 3.1. □

The next lemma considers some typical examples.

Lemma 3.6. *Let $2 \leq i \neq j \leq r+1$, and let $\lambda = -2xy$. Then:*

(i) *If $\alpha = [E(e_i)(x), E(e_j)(y)]$ and $\alpha^* = [E(e_j)(-y), E(e_i)(-x)]$, then $\alpha^* = \alpha^{-1}$, and $S_r(vC_{i-1j-1}(\lambda), w) = \alpha S_r(v,w)\alpha^{-1}$.*

(ii) *If $\beta = [E(e_i^*)(x), E(e_j^*)(y)]$ and $\beta^* = [E(e_j^*)(-y), E(e_i^*)(-x)]$, then $\beta^* = \beta^{-1}$, and $S_r(v, wC_{i-1j-1}(\lambda)) = \beta S_r(v,w)\beta^{-1}$.*

(iii) *If $\gamma = [E(e_j)(x), E(e_i^*)(y)]$ and $\gamma^* = [E(e_i^*)(-y), E(e_j)(-x)]$, then $\gamma^* = \gamma^{-1}$, and $S_r(vE_{ij}(\lambda), wE_{ji}(-\lambda)) = \gamma S_r(v,w)\gamma^{-1}$.*

Proof. The reader may consider the following approaches to prove the above lemma: Directly verify via Lemma 3.2, or do direct computations with the matrices and use observations as in Lemma 3.1. □

4. Commutator laws in $EUm_r(R)^*$

We next state and prove the "commutator laws" in $EUm_r(R)^*$. The basic instinct of these formulae is to convey the information that the commutator of an elementary unimodular generator in $\mathbb{Z}[\lambda]$, and an elementary unimodular generator in $\mathbb{Z}[\mu]$, will be an elementary unimodular generator in $\mathbb{Z}[\lambda\mu]$.

Lemma 4.1. *For $1 < i < j \leq r+1$, $\lambda, \mu \in R$, $c_i = e_i$ or e_i^*, $d_j = e_j$ or e_j^*, we have, for $r \geq 2$,*

$$\begin{aligned}&[E_r(c_i)(\lambda), E_r(d_j)(\mu)]\\ &\quad= [E_{r-1}(c_{i-1})(\lambda), E_{r-1}(d_{j-1})(\mu)] \perp [E_{r-1}(c_{i-1})(\lambda), E_{r-1}(d_{j-1})(\mu)]\\ &\quad= \alpha \perp \cdots \perp \alpha,\end{aligned}$$

where

$$\alpha = \begin{cases} \{E_{r-i+1}(d_{j-i+1})(2\lambda\mu)^{top} \perp E_{r-i+1}(d_{j-i+1})(-2\lambda\mu)^{bot}\} & \text{if } c_i = e_i,\\ \{E_{r-i+1}(d_{j-i+1})(2\lambda\mu)^{bot} \perp E_{r-i+1}(d_{j-i+1})(-2\lambda\mu)^{top}\} & \text{if } c_i = e_i^*.\end{cases}$$

(The α is taken 2^{i-2} times above.)

Proof. We only prove here the first result when $c_i = e_i$, and $d_j = e_j$; the rest are proved similarly. We do it by induction on i. Note that

$$\begin{aligned}&[E_r(e_i)(\lambda), E_r(e_j)(\mu)]\\ &= \left[\begin{pmatrix} I & S_{r-1}(\lambda e_{i-1},0)\\ -S_{r-1}(0,\lambda e_{i-1})^T & I\end{pmatrix}, \begin{pmatrix} I & S_{r-1}(\mu e_{j-1},0)\\ -S_{r-1}(0,\mu e_{j-1})^T & I\end{pmatrix}\right]\\ &= \begin{pmatrix} I - S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T & S_{r-1}(\lambda e_{i-1}+\mu e_{j-1},0)\\ -S_{r-1}(0,\lambda e_{i-1}+\mu e_{j-1})^T & I - S_{r-1}(0,\lambda e_{i-1})^T S_{r-1}(\mu e_{j-1},0)\end{pmatrix}\\ &\times \begin{pmatrix} I - S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T & -S_{r-1}(\lambda e_{i-1}+\mu e_{j-1},0)\\ S_{r-1}(0,\lambda e_{i-1}+\mu e_{j-1})^T & I - S_{r-1}(0,\lambda e_{i-1})^T S_{r-1}(\mu e_{j-1},0)\end{pmatrix}.\end{aligned}$$

Via Lemma 3.1,

(i) $S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T$

$$\begin{aligned}&= -S_{r-1}(\mu e_{j-1},0)S_{r-1}(0,\lambda e_{i-1})^T S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T\\ &= 0.\end{aligned}$$

(ii) $S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T S_{r-1}(\lambda e_{i-1}+\mu e_{j-1},0)$

$$\begin{aligned} &= S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T[S_{r-1}(\lambda e_{i-1},0)+S_{r-1}(\mu e_{j-1},0)] \\ &= S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T S_{r-1}(\lambda e_{i-1},0) \\ &= -S_{r-1}(\mu e_{j-1},0)S_{r-1}(0,\lambda e_{i-1})^T S_{r-1}(\lambda e_{i-1},0)=0. \end{aligned}$$

Hence,

$$[E_r(e_i)(\lambda),E_r(e_j)(\mu)] = \begin{pmatrix} I-2S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T & 0 \\ 0 & I-2S_{r-1}(0,\lambda e_{i-1})^T S_{r-1}(\mu e_{j-1},0)\end{pmatrix}.$$

When $i=2$,

$$\begin{aligned} &I-2S_{r-1}(\lambda e_1,0)S_{r-1}(0,\mu e_{j-1})^T \\ &\quad = I-2\begin{pmatrix}\lambda I_{2^{r-2}} & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & -S_{r-2}(\mu e_{j-2},0) \\ S_{r-2}(0,\mu e_{j-2})^T & 0\end{pmatrix} \\ &\quad = \begin{pmatrix} I_{2^{r-2}} & S_{r-2}(2\lambda\mu e_{j-2},0) \\ 0 & I_{2^{r-2}}\end{pmatrix} = E_{r-1}(e_{j-1})(2\lambda\mu)^{top}. \end{aligned}$$

Similarly, by direct computation, we can show that

$$I-2S_{r-1}(0,\lambda e_{i-1})^T S_{r-1}(\mu e_{j-1},0)=E_{r-1}(e_{j-1})(-2\lambda\mu)^{bot}.$$

Hence $[E_r(e_2)(\lambda),\, E_r(e_j)(\mu)] = E_{r-1}(e_{j-1})(2\lambda\mu)^{top} \perp E_{r-1}(e_{j-1})(-2\lambda\mu)^{bot}$.

Now consider the case when $i>2$. Via Lemma 3.1, we can write

$$[E_r(e_i)(\lambda),E_r(e_j)(\mu)] = \begin{pmatrix} I-2S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T & 0 \\ 0 & I-2S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T\end{pmatrix}.$$

Note that

$$\begin{aligned} &I-2S_{r-1}(\lambda e_{i-1},0)S_{r-1}(0,\mu e_{j-1})^T \\ &\quad = I-2\begin{pmatrix} 0 & S_{r-2}(\lambda e_{i-2},0) \\ -S_{r-2}(0,\lambda e_{i-2})^T & 0\end{pmatrix}\begin{pmatrix} 0 & -S_{r-2}(\mu e_{j-2},0) \\ S_{r-2}(0,\mu e_{j-2})^T & 0\end{pmatrix} \\ &\quad = \begin{pmatrix} I-2S_{r-2}(\lambda e_{i-2},0)S_{r-2}(0,\mu e_{j-2})^T & 0 \\ 0 & I-2S_{r-2}(0,\lambda e_{i-2})^T S_{r-2}(\mu e_{j-2},0)\end{pmatrix} \\ &\quad = [E_{r-1}(e_{i-1})(\lambda),E_{r-1}(e_{j-1})(\mu)]. \end{aligned}$$

Hence

$$\begin{aligned} &[E_r(e_i)(\lambda),E_r(e_j)(\mu)] \\ &\qquad = [E_{r-1}(e_{i-1})(\lambda),E_{r-1}(e_{j-1})(\mu)] \perp [E_{r-1}(e_{i-1})(\lambda),E_{r-1}(e_{j-1})(\mu)]. \end{aligned}$$

The rest follows by a simple induction argument on i. □

Corollary 4.2. *For $2 \le i \ne j \le r+1$,*

$$\begin{aligned} [E(e_i)(\lambda),E(e_j^*)(\mu)] &= [E(e_i)(x),E(e_j^*)(y)], \ \textit{if } xy=\lambda\mu, \\ [E(e_i)(\lambda),E(e_j)(\mu)] &= [E(e_i)(x),E(e_j)(y)], \ \textit{if } xy=\lambda\mu, \\ [E(e_i^*)(\lambda),E(e_j^*)(\mu)] &= [E(e_i^*)(x),E(e_j^*)(y)], \ \textit{if } xy=\lambda\mu. \end{aligned}$$

Proof. This is obvious from the previous result. □

The next lemma plays the role in $EUm_r(R)^*$ of the role played by the commutator law $E_{ij}(\lambda\mu) = [E_{ik}(\lambda), E_{kj}(\mu)]$, for $\lambda, \mu \in R$, $1 \le i \ne j \ne k \le r+1$, in calculations done in $E_{r+1}(R)$.

Lemma 4.3 (Biduality). *For $2 \le j \ne k \le r+1$,*

$$\begin{aligned} E_r(e_k^*)(2\lambda\mu\nu)[E(e_j)(\lambda), E(e_k^*)(\lambda\mu\nu)] &= [x,[y,z]], \\ E_r(e_k)(2\lambda\mu\nu)[E(e_j^*)(\lambda), E(e_k)(\lambda\mu\nu)] &= [x^*,[y^*,z^*]], \end{aligned}$$

where $x = E_r(e_j)(\lambda)$, $y = E_r(e_k^*)(\mu)$, $z = E_r(e_j^*)(\nu)$, $x^* = E_r(e_j^*)(\lambda)$, $y^* = E_r(e_k)(\mu)$, $z^* = E_r(e_j)(\nu)$.

Proof. We establish the first equation, and the second can be done similarly.

First we consider the case when $j = 2$. By Lemma 4.1, $[x,[y,z]] = [\alpha, \beta]$, where $\alpha = x$ and $\beta = E(e_{k-1}^*)(-2\mu\nu)^{bot} \perp E(e_{k-1}^*)(2\mu\nu)^{top}$.

Now by direct computation, we get $[\alpha,\beta] = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, where

$$\begin{aligned}
A_{11} &= I_{2^{r-1}} + S_{r-1}(\lambda e_1, 0)E(e_{k-1}^*)(2\mu\nu)^{top} S_{r-1}(0,\lambda e_1)^T E(e_{k-1}^*)(-2\mu\nu)^{bot} \\
&= \begin{pmatrix} I_{2^{r-2}} & S_{r-2}(0, 2\lambda^2\mu\nu e_{k-2}) \\ 0 & I_{2^{r-2}} \end{pmatrix} = E(e_{k-1}^*)(2\lambda^2\mu\nu)^{top}, \\
A_{12} &= -E(e_{k-1}^*)(-2\mu\nu)^{bot} S_{r-1}(\lambda e_1, 0)E(e_{k-1}^*)(2\mu\nu)^{top} + S_{r-1}(\lambda e_1, 0) \\
&= \begin{pmatrix} 0 & S_{r-2}(0, 2\lambda\mu\nu e_{k-2}) \\ -S_{r-2}(2\lambda\mu\nu e_{k-2}, 0)^T & 0 \end{pmatrix} = S_{r-1}(0, 2\lambda\mu\nu e_{k-1}) \\
&= S_{r-1}(0, 2\lambda\mu\nu e_{k-1}) E(e_{k-1}^*)(-2\lambda^2\mu\nu)^{bot}, \\
A_{21} &= E(e_{k-1}^*)(2\mu\nu)^{top} S_{r-1}(0,\lambda e_1)^T E(e_{k-1}^*)(-2\mu\nu)^{bot} - S_{r-1}(0,\lambda e_1)^T \\
&= \begin{pmatrix} 0 & S_{r-2}(0, 2\lambda\mu\nu e_{k-2}) \\ -S_{r-2}(2\lambda\mu\nu e_{k-2}, 0)^T & 0 \end{pmatrix} = S_{r-1}(0, 2\lambda\mu\nu e_{k-1}) \\
&= -S_{r-1}(2\lambda\mu\nu e_{k-1}, 0)^T = -S_{r-1}(2\lambda\mu\nu e_{k-1}, 0)^T E(e_{k-1}^*)(2\lambda^2\mu\nu)^{top}, \\
A_{22} &= I_{2^{r-1}} + S_{r-1}(0,\lambda e_1)^T E(e_{k-1}^*)(-2\mu\nu)^{bot} S_{r-1}(\lambda e_1, 0) E(e_{k-1}^*)(2\mu\nu)^{top} \\
&= \begin{pmatrix} I_{2^{r-2}} & 0 \\ S_{r-2}(2\lambda^2\mu\nu e_{k-2}, 0)^T & I_{2^{r-2}} \end{pmatrix} = E(e_{k-1}^*)(-2\lambda^2\mu\nu)^{bot}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&[x,[y,z]] \\
&= \begin{pmatrix} E(e_{k-1}^*)(2\lambda^2\mu\nu)^{top} & S_{r-1}(0, 2\lambda\mu\nu e_{k-1})E(e_{k-1}^*)(-2\lambda^2\mu\nu)^{bot} \\ -S_{r-1}(2\lambda\mu\nu e_{k-1}, 0)^T E(e_{k-1}^*)(2\lambda^2\mu\nu)^{top} & E(e_{k-1}^*)(-2\lambda^2\mu\nu)^{bot} \end{pmatrix} \\
&= E(e_k^*)(2\lambda\mu\nu)[E(e_2)(\lambda), E(e_k^*)(\lambda\mu\nu)],
\end{aligned}$$

as required. The last equalities in A_{12}, and A_{21} above follow by a direct computation to show that

$$\begin{aligned}
S_{r-1}(0, 2\lambda\mu\nu e_{k-1})E(e_{k-1}^*)(-2\lambda^2\mu\nu)^{bot} &= S_{r-1}(0, 2\lambda\mu\nu e_{k-1}) \\
-S_{r-1}(2\lambda\mu\nu e_{k-1}, 0)^T E(e_{k-1}^*)(2\lambda^2\mu\nu)^{top} &= S_{r-1}(0, 2\lambda\mu\nu e_{k-1}) \\
&= -S_{r-1}(2\lambda\mu\nu e_{k-1}, 0)^T,
\end{aligned}$$

as $k \ne 2$. The case when $k = 2$ can be done similarly as above.

Now we consider the general case when $2 < j \ne k \le r+1$.

By Lemma 4.1, $[x,[y,z]] = [\alpha,\beta]$, where $\alpha = x$, $\beta = A \perp A$, for $A = I - 2S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T$. Moreover, $[\alpha,\beta] = \gamma\delta$, where

$$\gamma = \begin{pmatrix} A & B \\ C & A \end{pmatrix}, \delta = \begin{pmatrix} A' & B' \\ C' & A' \end{pmatrix},$$

where $B = S_{r-1}(\lambda e_{j-1},0)A$, $C = -S_{r-1}(0,\lambda e_{j-1})^T A$, $B' = -S_{r-1}(\lambda e_{j-1},0)A'$, $C' = S_{r-1}(0,\lambda e_{j-1})^T A'$, $A' = I + 2S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T$.

Note via Lemma 3.1 that

$$\begin{aligned} & S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T S_{r-1}(\lambda e_{j-1},0) \\ = \quad & -S_{r-1}(0,\mu e_{k-1})[S_{r-1}(0,\lambda e_{j-1})^T S_{r-1}(0,\nu e_{j-1}) - \lambda\nu I] \\ = \quad & S_{r-1}(\lambda e_{j-1},0)S_{r-1}(\mu e_{k-1},0)^T S_{r-1}(0,\nu e_{j-1}) + \lambda\nu S_{r-1}(0,\mu e_{k-1}) \\ = \quad & S_{r-1}(\lambda e_{j-1},0)S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T + \lambda\nu S_{r-1}(0,\mu e_{k-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & S_{r-1}(\lambda e_{j-1},0)\{I - 2S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T\} \\ = \quad & \{I - 2S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T\}S_{r-1}(\lambda e_{j-1},0) - 2\lambda\nu S_{r-1}(\mu e_{k-1},0)^T. \end{aligned}$$

Hence for $\alpha = \{I - 2S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T\}$,

$$S_{r-1}(\lambda e_{j-1},0)\alpha = \alpha S_{r-1}(\lambda e_{j-1},0) + 2\lambda\nu S_{r-1}(0,\mu e_{k-1}).$$

Therefore, as $S_{r-1}(\lambda e_{j-1},0) = -S_{r-1}(0,\lambda e_{j-1})^T$,

$$-S_{r-1}(0,\lambda e_{j-1})^T\alpha = -\alpha S_{r-1}(0,\lambda e_{j-1})^T + 2\lambda\nu S_{r-1}(0,\mu e_{k-1}).$$

Also for $\beta = \{I + 2S_{r-1}(0,\mu e_{k-1})S_{r-1}(\nu e_{j-1},0)^T\}$, one can show that

$$\begin{aligned} -S_{r-1}(\lambda e_{j-1},0)\beta &= -\beta S_{r-1}(\lambda e_{j-1},0) + 2\lambda\nu S_{r-1}(0,\mu e_{k-1}), \\ S_{r-1}(0,\lambda e_{j-1})^T\beta &= \beta S_{r-1}(0,\lambda e_{j-1})^T + 2\lambda\nu S_{r-1}(0,\mu e_{k-1}). \end{aligned}$$

Thus, using the above identities, for $\gamma = 2S_{r-1}(\lambda\mu\nu e_{k-1},0)^T S_{r-1}(\lambda e_{j-1},0)$ one can show that

$$\begin{aligned} [x,[y,z]] &= \begin{pmatrix} I+\gamma & S_{r-1}(0,2\lambda\mu\nu e_{k-1}) \\ -S_{r-1}(2\lambda\mu\nu e_{k-1},0)^T & I+\gamma \end{pmatrix} \\ &= E(e_k^*)(2\lambda\mu\nu)[E(e_j)(\lambda),E(e_k^*)(\lambda\mu\nu)], \end{aligned}$$

as was claimed. □

5. Structure theorem for $EUm_r(R)$

The elementary subgroup $E_r(R)$ of $GL_\gamma(R)$ is generated by the basic generators $E_{ij}(\lambda)$, $1 \le i \ne j \le r$, $\lambda \in R$. In view of the commutator laws, the generators $E_{1i}(\lambda)$, $E_{i1}(\lambda)$, $2 \le i \le r$, $\lambda \in R$, suffices to generate $E_r(R)$.

Therefore, it seems natural to expect that the elementary unimodular generators $E(e_i)(\lambda)$, $2 \le i \le r+1$, $\lambda \in R$, should suffice to generate $EUm_r(R)$. However, this is not the case, as a simple example will show: Note that elementary generators of type $E_2(e_p)(\mu)$, $E_2(e_q)(\nu)$, are (elementary) symplectic. However, $S_2(e_2+e_1,e_2)$ is not symplectic, and so cannot be written as a product of these generators.

We examine below which additional elementary unimodular generators need to be added to the above set of generators, so as to suffice. The next two observations obviate the necessity of some generators.

Lemma 5.1. *Let R be a commutative ring. For $1 < i \neq j \leq r+1$, $\lambda \in R$, $S_r(e_j + \lambda e_i, e_j)$ can be written as a product of generators $E_r(c)(\lambda)$, for $c \in \{e_i, e_i^*, e_j, e_j^*\}$.*

Proof. By Lemma 3.2,

$$\begin{aligned} S_r(e_j, e_j) &= E(e_j^*)(1)^{bot} S_r(e_j, e_1 + e_j) E(e_j^*)(1)^{top} \\ &= E(e_j^*)(1) E(e_j^*)(-1)^{top} S_r(e_j, e_1 + e_j) E(e_j^*)(-1)^{bot} E(e_j^*)(1) \\ &= E(e_j^*)(1) S_r(e_1 + e_j, e_1) E(e_j^*)(1) \\ &= E(e_j)(1) S_r(e_1, e_1 + e_j) E(e_j)(1). \end{aligned}$$

By Lemma 3.6, $S_r(e_j+\lambda e_i, e_j) = S_r(e_j E_{ji}(\lambda), e_j E_{ij}(-\lambda))$ is a conjugate of $S_r(e_j, e_j)$ by means of above type of generators. □

Lemma 5.2. *Let R be a commutative ring in which 2 is invertible. For $\lambda, \mu \in R$, and for all r, $2 \leq i \neq j \leq r+1$, we have*

$$\begin{aligned} S_r(e_1, e_1 + \lambda e_i + \mu e_j) &= \alpha\beta\beta\alpha, \\ S_r(e_1 + \lambda e_i + \mu e_j, e_1) &= \gamma\delta\delta\gamma, \\ S_r(e_1 + \lambda e_i, e_1 + \mu e_j) &= \gamma\beta\beta\gamma, \end{aligned}$$

where $\alpha = S_r(e_1, e_1 + \frac{\lambda}{2} e_i)$, $\beta = S_r(e_1, e_1 + \frac{\mu}{2} e_j)$, $\gamma = S_r(e_1 + \frac{\lambda}{2} e_i, e_1)$, $\delta = S_r(e_1 + \frac{\mu}{2} e_j, e_1)$.

Proof. This follows from Lemma 3.2, or by a direct computation. □

Corollary 5.3. *Let R be a commutative ring in which 2 is invertible. For $\lambda, \mu \in R$, $2 \leq i \neq j \leq r+1$, $r \geq 2$ and $c = e_i$ or e_i^*, $d = e_j$ or e_j^*, one has*

$$[E(c)(\mu), E(d)(\lambda)^{tb}] = [E(c)(\frac{\mu}{2}), E(d)(\lambda)].$$

Proof. We prove one case; the other cases can be proved similarly. Via Lemma 3.2, Lemma 5.2,

$$\begin{aligned} &[E(e_j)(\mu), E(e_i^*)(\lambda)^{bot}] \\ &= E(e_j)(\mu) E(e_i^*)(\lambda)^{bot} E(e_j)(-\mu) E(e_i^*)(\lambda)^{bot^{-1}} \\ &= E(e_j)(\mu)\{E(e_i^*)(\lambda)^{bot} E(e_j)(-\mu) E(e_i^*)(\lambda)^{top}\} E(e_i^*)(-\lambda) \\ &= E(e_j)(\mu) S_r(e_1 - \mu e_j, e_1 + \lambda e_i) E(e_i^*)(-\lambda) \\ &= E(e_j)(\mu) E(e_j)(-\frac{\mu}{2}) E(e_i^*)(\lambda) E(e_j)(-\frac{\mu}{2}) E(e_i^*)(-\lambda) \\ &= [E(e_j)(\frac{\mu}{2}), E(e_i^*)(\lambda)]. \end{aligned}$$

□

Definition. For $2 \leq i \leq r+1$, $\lambda \in R$, let

$$\begin{aligned} E_r(e_{i1})(\lambda) &= S_r(e_i + \lambda e_1, e_i), \\ E_r(e_{i1}^*)(\lambda) &= S_r(e_i, e_i + \lambda e_1). \end{aligned}$$

(When the size is clear, the size subscript may be dropped.)

Remark. It is easy to verify that the generators $E(c)(\lambda) S_r(e_i, e_i)^{-1}$, $c = e_{i1}$ or e_{i1}^*, $\lambda \in R$, satisfy the splitting property

$$E(c)(\lambda + \mu) S_r(e_i, e_i)^{-1} = E(c)(\lambda) S_r(e_i, e_i)^{-1}. E(c)(\mu) S_r(e_i, e_i)^{-1}.$$

Lemma 5.4. *Let R be a commutative ring in which 2 is invertible. For $2 \leq i \neq j \leq r+1$, $\lambda, \mu \in R$, $S_r(e_1 + \lambda e_j, (1-\lambda\mu)e_1 + \mu e_j)$ can be written as a product of generators of type $E_r(c)(a)$, for some $a \in R$, $c = e_i$ or e_i^*, e_{i1} or e_{i1}^*.*

Proof. By Lemma 3.2, one can check that

$$S_r((1+\lambda)e_1 - \lambda e_i, e_1 + e_i) = E(e_i)(-1)E(e_{i1})(\lambda)E(e_i)(-1).$$

(The above is a special case. For instance, the case when $\mu = -\lambda$, $\lambda = 1$, is the transpose inverse of this case.) In general, by Lemma 3.2,

$$\begin{aligned} & S_r(e_1 + \lambda e_j, (1-\lambda\mu)e_1 + \mu e_j) \\ = \;& E(e_j)(\lambda)^{bot} E(e_j^*)(\mu) E(e_j)(\lambda)^{top} \\ = \;& E(e_j)(\lambda)^{bot}[x,[y,z]][E(e_j^*)(\frac{\mu}{2}), E(e_k)(\frac{\mu}{2})]E(e_j)(\lambda)^{top} \end{aligned}$$

by biduality, where $x = E(e_k)(\frac{\mu}{2})$, $y = E(e_j^*)(1)$, $z = E(e_k^*)(1)$, for some $k \neq j$.

The three cases which occur are similar to

$$\begin{aligned} & E(e_j)(\lambda)^b E(e_k)(\mu) E(e_j)(\lambda)^t, \\ & E(e_j)(\lambda)^b E(e_j^*)(1) E(e_j)(\lambda)^t, \\ & E(e_j)(\lambda)^b E(e_k^*)(1) E(e_j)(\lambda)^t, \end{aligned}$$

and are easily covered by applying Lemma 3.2, Corollary 5.3, and the above special case. □

Lemma 5.5. *Let R be a commutative ring with 1. For $2 \leq i \neq j \leq r+1$,*

(i) $[[E_r(e_i^*)(x), E_r(e_j^*)(y)], E_r(e_j)(z)^{top}] = E_r(e_i^*)(-2xyz)^{top}$.
(ii) $[[E_r(e_i)(x), E_r(e_j)(y)], E_r(e_j^*)(z)^{top}] = E_r(e_i)(-2xyz)^{top}$.
(iii) $[[E_r(e_i)(x), E_r(e_j^*)(y)], E_r(e_j)(z)^{top}] = E_r(e_i)(-2xyz)^{top}$.
(iv) $[[E_r(e_i^*)(x), E_r(e_j)(y)], E_r(e_j^*)(z)^{top}] = E_r(e_i^*)(-2xyz)^{top}$.
(v) $[E_r(e_j)(-z)^{bot}, [E_r(e_i^*)(x), E_r(e_j^*)(y)]] = E_r(e_i^*)(-2xyz)^{bot}$.
(vi) $[E_r(e_j)(-z)^{bot}, [E_r(e_i^*)(x), E_r(e_j)(y)]] = E_r(e_i)(-2xyz)^{bot}$.
(vii) $[E_r(e_j^*)(-z)^{bot}, [E_r(e_i)(x), E_r(e_j)(y)]] = E_r(e_i)(-2xyz)^{bot}$.
(viii) $[E_r(e_j^*)(-z)^{bot}, [E_r(e_i)(x), E_r(e_j^*)(y)]] = E_r(e_i^*)(-2xyz)^{bot}$.

Proof. We prove (i); the rest are proved similarly. First we consider the case when $i = 2$. By Lemma 4.1,

$$[E(e_2^*)(x), E(e_j^*)(y)] = \begin{pmatrix} E(e_{j-1}^*)(2xy)^{bot} & 0 \\ 0 & E(e_{j-1}^*)(-2xy)^{top} \end{pmatrix}.$$

Hence for $P = E(e_{j-1}^*)(2xy)^{bot}$ and $Q = E(e_{j-1}^*)(-2xy)^{top}$, we have

$$\begin{aligned} & [[E(e_2^*)(x), E(e_j^*)(y)], E(e_j)(z)^{top}] \\ & \quad = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I & S_{r-1}(ze_{j-1}, 0) \\ 0 & I \end{pmatrix} \begin{pmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} I & -S_{r-1}(ze_{j-1}, 0) \\ 0 & I \end{pmatrix} \\ & \quad = \begin{pmatrix} I & -S_{r-1}(ze_{j-1}, 0) + PS_{r-1}(ze_{j-1}, 0)Q^{-1} \\ 0 & I \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned}
&- S_{r-1}(ze_{j-1},0) + PS_{r-1}(ze_{j-1},0)Q^{-1} \\
&\quad = -S_{r-1}(ze_{j-1},0) + \begin{pmatrix} 0 & S_{r-2}(ze_{j-2},0) \\ -S_{r-2}(0,ze_{j-2})^T & -2xyzI_{2^{r-2}} \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 & 0 \\ 0 & -2xyzI_{2^{r-2}} \end{pmatrix} = S_{r-1}(0,-2xyze_1).
\end{aligned}$$

Thus

$$\begin{aligned}
[[E(e_2^*)(x), E(e_j^*)(y)], E(e_j)(z)^{top}] &= \begin{pmatrix} I_{2^{r-1}} & S_{r-1}(0,-2xyze_1) \\ 0 & I_{2^{r-1}} \end{pmatrix} \\
&= E(e_2^*)(-2xyz)^{top},
\end{aligned}$$

as required. We now consider the case when $j = 2$. By Lemma 4.1,

$$[E(e_i^*)(x), E(e_2^*)(y)] = \begin{pmatrix} E(e_{i-1}^*)(-2xy)^{bot} & 0 \\ 0 & E(e_{i-1}^*)(2xy)^{top} \end{pmatrix}.$$

Hence for $P = E(e_{i-1}^*)(-2xy)^{bot}$ and $Q = E(e_{i-1}^*)(2xy)^{top}$, we have

$$\begin{aligned}
&[[E(e_i^*)(x), E(e_2^*)(y)], E(e_2)(z)^{top}] \\
&\quad = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I & S_{r-1}(ze_1,0) \\ 0 & I \end{pmatrix} \begin{pmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} I & -S_{r-1}(ze_1,0) \\ 0 & I \end{pmatrix} \\
&\quad = \begin{pmatrix} I & -S_{r-1}(ze_1,0) + PS_{r-1}(ze_1,0)Q^{-1} \\ 0 & I \end{pmatrix}.
\end{aligned}$$

Now

$$\begin{aligned}
&- S_{r-1}(ze_1,0) + PS_{r-1}(ze_1,0)Q^{-1} \\
&\quad = -S_{r-1}(ze_1,0) + \begin{pmatrix} zI_{2^{r-2}} & -S_{r-2}(0,2xyze_{i-2}) \\ S_{r-2}(2xyze_{i-2},0)^T & 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 & -S_{r-2}(0,2xyze_{i-2}) \\ S_{r-2}(2xyze_{i-2},0)^T & 0 \end{pmatrix} = S_{r-1}(0,-2xyze_{i-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
[[E(e_i^*)(x), E(e_2^*)(y)], E(e_2)(z)^{top}] &= \begin{pmatrix} I_{2^{r-1}} & S_{r-1}(0,-2xyze_{i-1}) \\ 0 & I_{2^{r-1}} \end{pmatrix} \\
&= E(e_i^*)(-2xyz)^{top},
\end{aligned}$$

as required.

Now we consider the general the case when $2 < i \neq j \leq r+1$. As noted in the proof of Lemma 4.1,

$$[E_r(e_i^*)(x), E_r(e_j^*)(y)] = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

where $A = I - 2S_{r-1}(0, xe_{i-1})S_{r-1}(ye_{j-1}, 0)^T$. Hence,

$$\begin{aligned}
&[[E_r(e_i^*)(x), E_r(e_j^*)(y)], E_r(e_j)^{top}(z)] \\
&\quad = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & S_{r-1}(ze_{j-1}, 0) \\ 0 & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & -S_{r-1}(ze_{j-1}, 0) \\ 0 & I \end{pmatrix} \\
&\quad = \begin{pmatrix} A & AS_{r-1}(ze_{j-1}, 0) \\ 0 & A \end{pmatrix} \begin{pmatrix} B & -BS_{r-1}(ze_{j-1}, 0) \\ 0 & B \end{pmatrix} \\
&\quad = \begin{pmatrix} I & -S_{r-1}(ze_{j-1}, 0) + AS_{r-1}(ze_{j-1}, 0)B \\ 0 & I \end{pmatrix},
\end{aligned}$$

for $B = I + 2S_{r-1}(0, xe_{i-1})S_{r-1}(ye_{j-1}, 0)^T$. As in the proof of Lemma 4.3,

$$\begin{aligned}
&- S_{r-1}(ze_{j-1}, 0) + AS_{r-1}(ze_{j-1}, 0)B \\
&\quad = -S_{r-1}(ze_{j-1}, 0) + A\{BS_{r-1}(ze_{j-1}, 0) - 2yzS_{r-1}(0, xe_{i-1})\} \\
&\quad = AS_{r-1}(0, -2xyze_{i-1}) \\
&\quad = \{I - 2S_{r-1}(0, xe_{i-1})S_{r-1}(ye_{j-1}, 0)^T\}S_{r-1}(0, -2xyze_{i-1}) \\
&\quad = S_{r-1}(0, -2xyze_{i-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
[[E_r(e_i^*)(x), E_r(e_j^*)(y)], E_r(e_j)^{top}(z)] &= \begin{pmatrix} I & S_{r-1}(0, -2xyze_{i-1}) \\ 0 & I \end{pmatrix} \\
&= E_r(e_i^*)(-2xyz)^{top},
\end{aligned}$$

as required. □

Proposition 5.6. *Let R be a commutative ring in which 2 is invertible. Then $EUm_r(R)^{tb} = EUm_r(R)$.*

Proof. $EUm_r(R)$ is generated by the Suslin matrices $S_r(v, w)$, with $v \in e_1 E_{r+1}(R)$, and with w such that $\langle v, w \rangle = 1$. As before, there exists $\varepsilon_1 \in E_{r+1}(R)$ such that $v_1 = e_1\varepsilon_1$, $w_1 = e_1\varepsilon_1^{T^{-1}}$. Therefore, by our key lemma, Lemma 3.2, $EUm_r(R) \subset EUm_r(R)^{tb}$. (This part does not need the fact that 2 is invertible in R.)

For the converse part, since 2 is invertible in R, it suffices to show that $E_r(e_i^*)(-2xyz)^{top}, E_r(e_i^*)(-2xyz)^{bot} \in EUm_r(R)$. By Lemma 5.5, we have

$$\begin{aligned}
E_r(e_i^*)(-2xyz)^{top} &= [[E_r(e_i^*)(x), E_r(e_j^*)(y)], E_r(e_j)(z)^{top}] \\
&= \{[E_r(e_i^*)(x), E_r(e_j^*)(y)]\} \\
&\quad \{E_r(e_j)(z)^{top}E_r(e_j^*)(y)E_r(e_j)(z)^{bot}\}\{E_r(e_j)(z)^{-1}\} \\
&\quad \{E_r(e_j)(z)^{top}E_r(e_i^*)(x)E_r(e_j)(z)^{bot}\}\{E_r(e_j)(z)^{-1}\} \\
&\quad \{E_r(e_j)(z)^{top}E_r(e_j^*)(y)^{-1}E_r(e_j)(z)^{bot}\}\{E_r(e_j)(z)^{-1}\} \\
&\quad \{E_r(e_j)(z)^{top}E_r(e_i^*)(x)^{-1}E_r(e_j)(z)^{bot}\}\{E_r(e_j)(z)^{-1}\}.
\end{aligned}$$

By Lemma 3.2 and Lemma 5.2,

$$E_r(e_j)(z)^{top}E_r(e_i^*)(x)E_r(e_j)(z)^{bot} = S_r(e_1 + ze_j, e_1 + xe_i) = \alpha\beta\beta\alpha,$$

where $\alpha = E_r(e_j)(\frac{z}{2})$ and $\beta = E_r(e_i^*)(\frac{x}{2})$. Again by Lemma 3.2 and Lemma 5.4, we have

$$E_r(e_j)(z)^{top}E_r(e_j^*)(y)E_r(e_j)(z)^{bot} = S_r((1 - yz)e_1 + ze_j, e_1 + ye_j) \in EUm_r(R).$$

Therefore,

$$\begin{aligned}
&E(e_i^*)(-2xyz)^{top}\\
&\quad = \{E(e_i^*)(x)\}\{E(e_j^*)(y)\}\{E(e_i^*)(x)^{-1}\}\{E(e_j^*)(y)^{-1}\}\\
&\qquad \{S_r((1-yz)e_1+ze_j, e_1+ye_j)\}\{E(e_j)(z)^{-1}\}\\
&\qquad \{E(e_j)(\frac{z}{2})\}\{E(e_i^*)(\frac{x}{2})\}\{E(e_i^*)(\frac{x}{2})\}\{E(e_j)(\frac{z}{2})\}\\
&\qquad \{E(e_j)(z)^{-1}\}\{S_r((1+yz)e_1+ze_j, e_1-ye_j)\}\{E(e_j)(z)^{-1}\}\\
&\qquad \{E(e_j)(\frac{z}{2})\}\{E(e_i^*)(\frac{x}{2})^{-1}\}\{E(e_i^*)(\frac{x}{2})^{-1}\}\{E(e_j)(\frac{z}{2})\}\{E(e_j)(z)^{-1}\}.
\end{aligned}$$

Hence, $E_r(e_i^*)(-2xyz)^{top} \in EUm_r(R)$.

We now show that "reversing above product" gives $E_r(e_i^*)(-2xyz)^{bot}$:

$$\begin{aligned}
&\{E_r(e_j)(z)^{-1}\}\{E_r(e_j)(\frac{z}{2})\}\{E_r(e_i^*)(\frac{x}{2})^{-1}\}\{E_r(e_i^*)(\frac{x}{2})^{-1}\}\{E_r(e_j)(\frac{z}{2})\}\\
&\{E_r(e_j)(z)^{-1}\}\{S_r((1+yz)e_1+ze_j, e_1-ye_j)\}\\
&\{E_r(e_j)(z)^{-1}\}\{E_r(e_j)(\frac{z}{2})\}\{E_r(e_i^*)(\frac{x}{2})\}\{E_r(e_i^*)(\frac{x}{2})\}\{E_r(e_j)(\frac{z}{2})\}\\
&\{E_r(e_j)(z)^{-1}\}\{S_r((1-yz)e_1+ze_j, e_1+ye_j)\}\\
&\{E_r(e_j^*)(y)^{-1}\}\{E_r(e_i^*)(x)^{-1}\}\{E_r(e_j^*)(y)\}\{E_r(e_i^*)(x)\}\\
&= \{E_r(e_j)(z)^{-1}\}\{E_r(e_j)(z)^tE_r(e_i^*)(x)^{-1}E_r(e_j)(z)^b\}\\
&\{E_r(e_j)(z)^{-1}\}\{E_r(e_j)(z)^tE_r(e_j^*)(y)^{-1}E_r(e_j)(z)^b\}\{E_r(e_j)(z)^{-1}\}\\
&\{E_r(e_j)(z)^tE_r(e_i^*)(x)E_r(e_j)(z)^b\}\{E_r(e_j)(z)^{-1}\}\\
&\{E_r(e_j)(z)^tE_r(e_j^*)(y)E_r(e_j)(z)^b\}\{[E_r(e_i^*)(x), E_r(e_j^*)(y)]^{-1}\}\\
&= E_r(e_j)(-z)^b[E_r(e_i^*)(x)^{-1}, E_r(e_j^*)(y)^{-1}]E_r(e_j)(z)^b[E_r(e_i^*)(x), E_r(e_j^*)(y)]^{-1}\\
&= E_r(e_j)(-z)^b[E_r(e_i^*)(-x), E_r(e_j^*)(-y)]E_r(e_j)(z)^b[E_r(e_i^*)(x), E_r(e_j^*)(y)]^{-1}\\
&= E_r(e_j)(-z)^b[E_r(e_j^*)(y), E_r(e_i^*)(x)]E_r(e_j)(z)^b[E_r(e_i^*)(x), E_r(e_j^*)(y)]^{-1}\\
&= [E_r(e_j)(-z)^b, [E_r(e_i^*)(x), E_r(e_j^*)(y)]] = E_r(e_i^*)(-2xyz)^b. \qquad \square
\end{aligned}$$

Theorem 5.7. *Let R be a commutative ring in which 2 is invertible. Let $\varepsilon \in E_{r+1}(R)$. Then $S_r(v\varepsilon, w\varepsilon^{T^{-1}}) \in H$, where H is the subgroup of $EUm_r(R)$ generated by elements of the type $E_r(c)(\lambda)$, for $c = e_i$, e_i^*, e_{i1}, or e_{i1}^*. In fact, $S_r(v\varepsilon, w\varepsilon^{T^{-1}}) = \alpha S_r(v,w)\alpha^*$, where $\alpha = S_1S_2\cdots S_n$, $\alpha^* = S_nS_{n-1}\cdots S_1$, where each S_i is of the type $E_r(c)(\lambda)$, for $c = e_i$, e_i^*, e_{i1}, or e_{i1}^**

Proof. By our key lemma, one can find $\alpha \in EUm_r(R)^{tb}$, $\alpha^* \in EUm_r(R)^{bt}$ (obtained by reversing the order and replacing *top* by *bottom*) such that $S_r(v\varepsilon, w\varepsilon^{T^{-1}}) = \alpha S_r(v,w)\alpha^*$. Now apply Proposition 5.6. The proof of Proposition 5.6 gives a procedure of how we can write α as a product of required type above, so that α^* is obtained by reversing the order of the product describing α. $\square$

In view of the above lemma one gets the structure theorem for the Elementary Unimodular vector group $EUm_r(R)$, when 2 is invertible in R:

Theorem 5.8. *Let R be a commutative ring in which 2 is invertible. Then $EUm_r(R)$ is generated by elements of the form $E(e_i)(w)$, $E(e_i^*)(x)$, $E(e_{i1})(y)$, $E(e_{i1}^*)(z)$, for $w, x, y, z \in R$, $2 \le i \le r+1$.* $\square$

Remark 5.9. One can show that $EUm_r(R) = EUm_r(R)^{tb}$, for any commutative ring R, without any additional assumption (cf. [1, Proposition 2.6]). However, it is not clear to us, when $2R \neq R$, whether the set of generators mentioned in Theorem 5.8 will suffice to generate $EUm_r(R)$.

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